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BIHARMONIC SUBMANIFOLDS IN SPHERES

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ABSTRACT

We give some methods to construct examples of nonharmonic biharmonic submanifolds of the unit n-dimensional sphere $\mathbb{S}^n$. In the case of curves in $\mathbb{S}^n$ we solve explicitly the biharmonic equation.

1. Introduction

Harmonic maps $\phi\colon (M,g) \to (N,h)$ between Riemannian manifolds are the critical points of the energy $E(\phi) = \frac{1}{2}\int_M |d\phi|^2\, v_g$, and they are therefore the solutions of the corresponding Euler–Lagrange equation for the energy. This equation is given by the vanishing of the tension field $\tau(\phi) = \operatorname{trace} \nabla d\phi$. As suggested by J. Eells and J. H. Sampson in [6], we can define the **bienergy** of a map ϕ by

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$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$, and say that ϕ is **biharmonic** if it is a critical point of the bienergy.

In [9, 10] G. Y. Jiang derived the first variation formula of the bienergy showing that the Euler–Lagrange equation for E_2 is

$$\tau_2(\phi) = J(\tau(\phi)) = 0, \tag{1.1}$$

where J is the Jacobi operator of ϕ. The equation $\tau_2(\phi) = 0$ is called the **biharmonic equation**.

In a different setting, in [2], B. Y. Chen defined biharmonic submanifolds of the Euclidean space as those with harmonic mean curvature vector, and stated the conjecture that any biharmonic submanifold of $\mathbb{R}^n$ is harmonic. As yet the conjecture has not been either proved or disproved, although some positive answers are known (see, for example, [5, 8]).

If we consider the biharmonic equation $\tau_2(\phi) = 0$ for isometric immersions into the Euclidean space we recover Chen's notion of biharmonic submanifolds, so the two definitions agree.

More generally, for biharmonic maps in a manifold with nonpositive sectional curvature, some nonexistence theorems have been already proved. For example, in [10], G.Y. Jiang proved that any biharmonic map from a compact orientable manifold to a manifold with nonpositive sectional curvature is harmonic. In the case of submanifolds, the third author, in [12], has proved that any biharmonic submanifold with constant mean curvature in a manifold with nonpositive sectional curvature is harmonic, i.e., minimal.

In the case $\dim N = \dim M + 1$ the above results of G. Y. Jiang and the third author are still true with the weaker assumption that the Ricci curvature in nonpositive [12].

The first part of this paper is devoted to proving some new results of nonexistence of nonharmonic biharmonic maps to a manifold with constant negative sectional curvature.

Next, we consider the problem of finding examples of nonharmonic biharmonic submanifolds of a manifold with *positive* sectional curvature. The case of $\mathbb{S}^3$ has been studied in [1], where the authors have given the classification of nonharmonic biharmonic submanifolds. They are: circles, spherical helices and parallel spheres.

The goal of this paper is to study nonharmonic biharmonic submanifolds of $\mathbb{S}^n$, for $n > 3$. In this case the family of such submanifolds is much larger. In fact, any minimal submanifold of a certain parallel hypersphere of $\mathbb{S}^n$ is a nonharmonic biharmonic submanifold of $\mathbb{S}^n$ (Theorem 3.5). Therefore, by using known minimal submanifolds, we can produce a large class of nonharmonic biharmonic submanifolds. For example, as a consequence of a well known result of Lawson ([11]), it turns out that there exist closed orientable nonminimal biharmonic surfaces of arbitrary genus in $\mathbb{S}^4$. On the other hand, the minimal Veronese embedding of $P^2(\mathbb{R})$ in $\mathbb{S}^4$ produces a nonorientable nonminimal biharmonic submanifold in $\mathbb{S}^5$.

In the last section we write down explicitly and solve the biharmonic equation for curves in $\mathbb{S}^n$.

NOTATION. We shall place ourselves in the C^∞ category, i.e., manifolds, metrics, connections, maps will be assumed to be smooth. By (M^m, g) we shall indicate a connected manifold of dimension m, without boundary, endowed with a Riemannian metric g. We shall denote by ∇ the Levi-Civita connection on (M, g). For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. The indices i, j, k, l take the values $1, 2, \ldots, m$.

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2. Nonexistence theorems

Let ϕ: $(M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. The tension field of ϕ is given by $\tau(\phi) = \operatorname{trace} \nabla d\phi$, and, for any compact domain $\Omega \subseteq M$, the **bienergy** is defined by

$$E_2(\phi) = \frac{1}{2} \int_\Omega |\tau(\phi)|^2 v_g.$$

Then we call **biharmonic** a smooth map ϕ which is a critical point of the bienergy functional for any compact domain $\Omega \subseteq M$. As we said in the introduction, we have for the bienergy the following first variation formula:

$$\left.\frac{dE_2(\phi_t)}{dt}\right|_{t=0} = \int_\Omega < \tau_2(\phi), V > v_g,$$

where v_g is the volume element, while V is the variational vector field along ϕ, and

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace}\ R^N(d\phi, \tau(\phi))d\phi. \tag{2.1}$$

Using a standard argument of the theory of strongly elliptic operators (see, for example, [13]), we have

PROPOSITION 2.1: *Let $\phi\colon (M,g) \to (N,h)$ be a biharmonic map. If ϕ is harmonic on an open subset U of M, then ϕ is harmonic.*

COROLLARY 2.2: *Let $\phi\colon (M,g) \to (N,h)$ be a biharmonic map. If ϕ is constant on an open subset U of M, then ϕ is constant.*

From now on, in this section, we take as N a manifold $N(-1)$ of constant negative sectional curvature -1 and we consider the canonical inclusion $\mathbf{i}\colon M \to N(-1)$ of a submanifold M of N.

In order to study the biharmonicity of $\mathbf{i}$, first of all we denote by B the second fundamental form, by A the shape operator, by H the mean curvature vector field of M in $N(-1)$, while $\nabla^{\perp}$ is the normal connection and $\Delta^{\perp}$ is the Laplacian in the normal bundle of M. Then we have

THEOREM 2.3: *The inclusion map $\mathbf{i}\colon M \to N(-1)$ is biharmonic if and only if*

$$\begin{cases} -\Delta^{\perp}H - \operatorname{trace} B(-, A_H -) - mH = 0, \\ 2\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) + \frac{m}{2}\operatorname{grad}(|H|^2) = 0. \end{cases} \tag{2.2}$$

Proof: Since

$$\operatorname{trace} R^N(d\mathbf{i}, \tau(\mathbf{i}))d\mathbf{i} = m\tau(\mathbf{i}),$$

the map $\mathbf{i}$ is biharmonic if and only if

$$\tau_2(\mathbf{i}) = \operatorname{trace}\nabla d\tau(\mathbf{i}) - m\tau(\mathbf{i}) = m\{\operatorname{trace}\nabla dH - mH\} = 0. \tag{2.3}$$

By a straightforward computation we obtain

$$\operatorname{trace}\nabla dH = -\Delta^{\perp}H - \operatorname{trace} B(-, A_H -) - 2\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) - \frac{m}{2}\operatorname{grad}(|H|^2).$$

Therefore, by replacing the value of $\operatorname{trace}\nabla dH$ in (2.3), we have that $\mathbf{i}$ is biharmonic if and only if

$$-\Delta^{\perp}H - \operatorname{trace} B(-, A_H -) - mH = 2\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) + \frac{m}{2}\operatorname{grad}(|H|^2). \tag{2.4}$$

Since the left-hand side of (2.4) is normal to M and the right-hand side of (2.4) is tangent, the theorem follows. ∎

We shall use the above theorem to prove that, in some cases, biharmonicity and harmonicity are equivalent.

We first consider pseudo-umbilical submanifolds, that is, submanifolds satisfying $A_H = |H|^2 I$. We have

THEOREM 2.4: *Let M be an m-dimensional pseudo-umbilical submanifold of $N(-1)$ with $m \neq 4$. Then M is biharmonic if and only if it is harmonic.*

Proof: Let $\{x^i\}_{i=1}^m$ be a system of normal coordinates around an arbitrary point $p \in M$, and let e_i be the corresponding coordinate vector fields. At p we have

$$\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) = \sum_i \nabla_{e_i} A_H(e_i) - \frac{m}{2}\operatorname{grad}(|H|^2).$$

Since M is pseudo-umbilical, the first term in the right-hand side is

$$\sum_i \nabla_{e_i} A_H(e_i) = \sum_i \nabla_{e_i}\left(|H|^2 e_i\right) = \sum_i e_i|H|^2 e_i = \operatorname{grad}(|H|^2),$$

and therefore

$$\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) = \left(1 - \frac{m}{2}\right)\operatorname{grad}(|H|^2). \tag{2.5}$$

Finally, substituting (2.5) in the second equation of (2.2) we obtain

$$(4 - m)\operatorname{grad}(|H|^2) = 0,$$

so, for $m \neq 4$, the mean curvature $|H|$ is constant. Since any biharmonic submanifold with constant mean curvature in a manifold with nonpositive sectional curvature is harmonic (see [12]), we have the theorem. ∎

In particular we have

COROLLARY 2.5: *Let γ: $I \to N(-1)$ be a curve parametrized by arc length. Then γ is biharmonic if and only if it is harmonic.*

In [3], B. Y. Chen and S. Ishikawa have proved that any biharmonic surface of the Euclidean 3-dimensional space is minimal, i.e., harmonic. The following theorem shows that the Chen–Ishikawa theorem remains true if we substitute the ordinary space with any 3-dimensional space with constant negative sectional curvature.

THEOREM 2.6: *Let M be a surface of $N^3(-1)$. Then M is biharmonic if and only if it is harmonic.*

Proof: Assume that M is a biharmonic submanifold. Suppose that M is non-harmonic. We shall prove that the mean curvature is constant, which means that

M is minimal. For this we shall follow closely the proof given by B. Y. Chen and S. Ishikawa, in [3], for biharmonic surfaces of $\mathbb{R}^3$.

Let $\{X_1, X_2\}$ be a local orthonormal frame field on M and let η be a unitary normal vector field. Assume that $H = f\eta$, where $f \in C^\infty(M)$ and $f > 0$. In this case conditions (2.2) become

$$\Delta f = (-2 - |A|^2)f, \tag{2.6}$$

$$A(\operatorname{grad} f) + f \operatorname{grad} f = 0. \tag{2.7}$$

Let $U = \{p \in M | (\operatorname{grad} f^2)(p) \neq 0\}$. We shall show that $U = \emptyset$.

Assume that $U \neq \emptyset$ and put

$$X_1 = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}.$$

We have

$$X_2 f = 0, \quad \operatorname{grad} f = (X_1 f)X_1, \tag{2.8}$$

and the second fundamental form B of M is given by

$$B(X_1, X_1) = -f\eta, \quad B(X_1, X_2) = 0, \quad B(X_2, X_2) = 3f\eta, \tag{2.9}$$

so

$$|A|^2 = 10f^2. \tag{2.10}$$

Since $N^3(-1)$ has constant sectional curvature and M is a hypersurface, the Codazzi equation gives

$$X_2 f = -4f\omega_2^1(X_1), \quad 3X_1 f = -4f\omega_1^2(X_2), \tag{2.11}$$

where $\{\omega^1, \omega^2\}$ are the 1-forms dual of $\{X_1, X_2\}$ and ω_i^j are the connection 1-forms given by $\nabla X_i = \omega_i^j X_j$. Now, (2.8) and (2.11) imply that $\omega_2^1(X_1) = 0$ and $d\omega^1 = 0$. Thus, locally, ω^1 is exact, that is, $\omega^1 = du$ for some function u. Since $df = (X_1 f)\omega^1 + (X_2 f)\omega^2$, and $X_2 f = 0$, we have that $df \wedge \omega^1 = 0$; this means that f is a function of u. Denoting by f' and f'' the first and second derivatives of f with respect to u, the second formula of (2.11) implies

$$4f\omega_1^2 = -3f'\omega^2. \tag{2.12}$$

Again, (2.8) and (2.11) give

$$4f\Delta f = 3(f')^2 - 4ff'', \tag{2.13}$$

and, from (2.6) and (2.10), we obtain

$$4ff'' - 3(f')^2 - 8f^2 - 40f^4 = 0. \tag{2.14}$$

If we put $(f')^2 = y$, condition (2.14) gives

$$2f\frac{dy}{df} - 3y = 40f^4 + 8f^2, \tag{2.15}$$

which implies

$$(f')^2 = 8f^4 + 8f^2 + Cf^{3/2}, \tag{2.16}$$

for some constant C.

On the other hand, the Gauss equation

$$K = -1 + \det A$$

gives

$$\begin{cases} K = -1 - 3f^2 \\ d\omega_1^2 = -K\omega^1 \wedge \omega^2 \end{cases} \tag{2.17}$$

where K is the Gaussian curvature of M. From (2.9), (2.12) and (2.17), we obtain

$$4ff'' - 7(f')^2 + 16f^4 + \frac{16}{3}f^2 = 0. \tag{2.15}$$

But (2.14) and (2.18) imply

$$(f')^2 = 14f^4 + \frac{10}{3}f^2. \tag{2.19}$$

Summing up, conditions (2.16) and (2.19) together say that f must satisfy a polynomial equation with constant coefficients, that is, f is constant. Hence, M has constant mean curvature. ∎

From Theorem 2.6 and Corollary 2.5 we have

THEOREM 2.7: *Let M be a submanifold of $N^3(-1)$. Then M is biharmonic if and only if it is harmonic.*

3. Biharmonic submanifolds of $\mathbb{S}^n$

The following example arises in the early works on biharmonic maps.

Example 3.1: [9, 10]. Let m_1, m_2 be two positive integers such that $m = m_1 + m_2$, and let r_1, r_2 be two positive real numbers such that $r_1^2 + r_2^2 = 1$. Then we have two cases:

1. $m_1 \neq m_2$, and $\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$ is a nonharmonic biharmonic submanifold of $\mathbb{S}^{m+1}$ if and only if $r_1 = r_2 = 1/\sqrt{2}$;
2. $m_1 = m_2 = q$, and the following statements are equivalent:
   - $\mathbb{S}^q(r_1) \times \mathbb{S}^q(r_2)$ is a biharmonic submanifold of $\mathbb{S}^{2q+1}$.
   - $\mathbb{S}^q(r_1) \times \mathbb{S}^q(r_2)$ is a harmonic submanifold of $\mathbb{S}^{2q+1}$.
   - $r_1 = r_2 = 1/\sqrt{2}$.

Note that in the case of $\mathbb{S}^3$ the above example gives the minimal Clifford torus: in fact, as mentioned in the introduction, the only nonminimal biharmonic surfaces of $\mathbb{S}^3$ are the parallel spheres of radius $1/\sqrt{2}$.

The next example was given by the authors in [1].

Example 3.2: Let $M = \mathbb{S}^m(a) \times \{b\} = \{p = (x^1, \ldots, x^{m+1}, b), |(x^1)^2 + \cdots + (x^{m+1})^2 = a^2,\ a^2 + b^2 = 1,\ 0 < a < 1\}$ be a parallel hypersphere of $\mathbb{S}^{m+1}$. Then $M = \mathbb{S}^m(a) \times \{b\}$ is a biharmonic submanifold of $\mathbb{S}^{m+1}$ if and only if $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$.

Note that the manifold $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 \neq m_2$, is not a pseudo-umbilical submanifold of $\mathbb{S}^{m+1}$, while the manifold $\mathbb{S}^m(a) \times \{b\}$ is pseudo-umbilical.

Except for these, so far we have not seen in the literature other examples of nonharmonic biharmonic submanifolds of the unit sphere $\mathbb{S}^n$.

In this section we propose some methods to construct new examples of biharmonic submanifolds. We first recall the following result.

THEOREM 3.3 ([12]): *Let M be a submanifold of $\mathbb{S}^n$ and let $\mathbf{i}\colon M \to \mathbb{S}^n$ be the canonical inclusion. Then the map $\mathbf{i}$ is biharmonic if and only if*

$$(3.1)\qquad \begin{cases} -\Delta^{\perp} H - \operatorname{trace} B(-, A_H -) + mH = 0, \\ 2 \operatorname{trace} A_{\nabla^{\perp}_{(-)} H}(-) + \frac{m}{2} \operatorname{grad}(|H|^2) = 0. \end{cases}$$

From Theorem 3.3, which is the analogue for $\mathbb{S}^n$ of Theorem 2.3, it follows immediately that a pseudo-umbilical submanifold of $\mathbb{S}^n$ whose mean curvature vector field is unitary and parallel is biharmonic. It is noteworthy that such a submanifold is actually harmonic in a hypersphere of $\mathbb{S}^n$, as shown in the following

THEOREM 3.4: *Let M be a pseudo-umbilical submanifold of $\mathbb{S}^n$ with mean vector field parallel and of norm equal to 1. Then*

1. *M is biharmonic in $\mathbb{S}^n$;*
2. *M is a minimal submanifold in a hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.*

Proof: We will use an idea of B. Y. Chen and K. Yano (see [4]). We denote by $\widetilde{H}$ the mean curvature vector field of M in $\mathbb{R}^{n+1}$. Then, for every $p \in M$, we have $\widetilde{H}(p) = H(p) - p$, and for any vector field X tangent to M,

$$\begin{aligned}\nabla_X^{\mathbb{R}^{n+1}} \widetilde{H} &= \widetilde{\nabla}_X^{\perp} \widetilde{H} - \widetilde{A}_{\widetilde{H}}(X) \\ &= (\nabla_X^{\mathbb{S}^n} H - < X, H > p) - \nabla_X^{\mathbb{R}^{n+1}} p \\ &= \nabla_X^{\perp} H - A_H(X) - X.\end{aligned}$$

Thus, by the hypothesis in the statement of the theorem, we have

$$\widetilde{\nabla}^{\perp} \widetilde{H} = \nabla^{\perp} H = 0 \quad \text{and} \quad \widetilde{A}_{\widetilde{H}} = 2I.$$

Now we consider the map $\Psi \in C^\infty(M; \mathbb{R}^{n+1})$ given by $\Psi(p) = p + \frac{1}{2}\widetilde{H}(p)$. We have

$$X(\Psi) = \nabla_X^{\mathbb{R}^{n+1}} \Psi = \nabla_X^{\mathbb{R}^{n+1}} p + \frac{1}{2} \nabla_X^{\mathbb{R}^{n+1}} \widetilde{H} = 0,$$

so Ψ is a constant vector.

Consequently

$$|p - \Psi|^2 = \frac{1}{4} |\widetilde{H}|^2 = \frac{1}{2}.$$

Thus $M \subset \mathbb{S}^n(\Psi; 1/\sqrt{2})$. Since $|\Psi| = 1/\sqrt{2}$, without loss of generality we can assume that $\Psi = (0, \ldots, 0, 1/\sqrt{2}) \in \mathbb{R}^{n+1}$; so $M \subset \mathbb{S}^n(\Psi; 1/\sqrt{2}) \cap \mathbb{S}^n = \mathbb{S}^{n-1}(1/\sqrt{2}) \times \{1/\sqrt{2}\}$.

Finally, since for every $p \in M$, the vector $(p - \Psi)$ is parallel to $\widetilde{H}(p)$, it follows that M is harmonic in $\mathbb{S}^n(\Psi; 1/\sqrt{2})$, and therefore, it is harmonic in $\mathbb{S}^{n-1}(1/\sqrt{2}) \times \{1/\sqrt{2}\}$. ∎

The last theorem suggests that in order to find nonharmonic biharmonic submanifolds of $\mathbb{S}^n$, we can search through harmonic submanifolds in hyperspheres. In fact we have the following

THEOREM 3.5: *Let M be a harmonic submanifold of $\mathbb{S}^n(a) \times \{b\}$, where $a^2 + b^2 = 1$, $0 < a < 1$. Then M is a nonharmonic biharmonic submanifold in $\mathbb{S}^{n+1}$ if and only if $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$.*

Proof: With respect to the standard Euclidean scalar product $<,>$ and to the rectangular coordinates (x^i) of $\mathbb{R}^{n+2}$, the set of sections of the tangent bundle of

$\mathbb{S}^n(a) \times \{b\}$ is

$$\{X = (X^1, \dots, X^{n+1}, 0) \in \mathbb{R}^{n+2} | x^1 X^1 + \cdots + x^{n+1} X^{n+1} = 0\}.$$

Let now $\xi = (x^1, \dots, x^{n+1}, -a^2/b)$ and $p = (x^1, \dots, x^{n+1}, b)$. Then we have

$$< \xi, X >= 0, \quad < \xi, p >= 0, \quad |\xi|^2 = a^2 + a^4/b^2 = c^2, \ c > 0,$$

and therefore ξ is a section of the normal bundle of $\mathbb{S}^n(a) \times \{b\}$ in $\mathbb{S}^{n+1}$. If we put $\eta = \frac{1}{c}\xi$ we have

$$\begin{aligned} \nabla^{\mathbb{S}^{n+1}}_X \eta =& \nabla^{\perp}_X \eta - A(X) \\ =& \frac{1}{c} \nabla^{\mathbb{S}^{n+1}}_X \xi = \frac{1}{c} \{ \nabla^{\mathbb{R}^{n+2}}_X \xi + < \xi, X > p \} \\ =& \frac{1}{c} \nabla^{\mathbb{R}^{n+2}}_{(X^1, \dots, X^{n+1}, 0)} (x^1, \dots, x^{n+1}, -a^2/b) \\ =& \frac{1}{c} X. \end{aligned}$$

This implies that $A = -\frac{1}{c} I$ and $\nabla^{\perp} \eta = 0$.

We denote by $\mathbf{i}$: $M \to \mathbb{S}^n(a) \times \{b\}$ and $\mathbf{i}_1$: $\mathbb{S}^n(a) \times \{b\} \to \mathbb{S}^{n+1}$ the inclusion maps. Let $\{X_i\}_{i=1}^m$ be a geodesic frame field around an arbitrary point $p \in M$. At p we have

$$\tau(\mathbf{i}_1 \circ \mathbf{i}) = \sum_{i=1}^m \nabla d\mathbf{i}_1(X_i, X_i) = \sum_i -\frac{1}{c} < X_i, X_i > \eta = -\frac{m}{c} \eta \neq 0,$$

and

$$\begin{aligned} \tau_2(\mathbf{i}_1 \circ \mathbf{i}) =& - \Delta \tau(\mathbf{i}_1 \circ \mathbf{i}) + m \tau(\mathbf{i}_1 \circ \mathbf{i}) \\ =& \sum_i \nabla^{\mathbb{S}^{n+1}}_{X_i} \nabla^{\mathbb{S}^{n+1}}_{X_i} \Big(- \frac{m}{c} \eta \Big) - \frac{m^2}{c} \eta \\ =& - \frac{m}{c} \sum_i \nabla^{\mathbb{S}^{n+1}}_{X_i} [\nabla^{\perp}_{X_i} \eta - A(X_i)] - \frac{m^2}{c} \eta \\ =& - \frac{m}{c^2} \sum_i \nabla^{\mathbb{S}^{n+1}}_{X_i} X_i - \frac{m^2}{c} \eta \\ =& \frac{m^2}{c} \Big(\frac{1}{c^2} - 1 \Big) \eta. \end{aligned}$$

Hence the composition cannot be harmonic and it is biharmonic if and only if $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$. ∎

Remark 3.6: Note that if M is harmonic in $\mathbb{S}^n(1/\sqrt{2})$, then it is automatically pseudo-umbilical in $\mathbb{S}^{n+1}$; moreover, $\nabla^\perp \tau(\mathbf{i}_1 \circ \mathbf{i}) = 0$ and $|\tau(\mathbf{i}_1 \circ \mathbf{i})| = m$.

Since the radial projection

$$\mathbb{S}^n \to \mathbb{S}^n(r) \quad x \mapsto rx, \quad r > 0,$$

is homothetic, all harmonic submanifolds in $\mathbb{S}^n$ become, after radial projection, harmonic submanifolds in $\mathbb{S}^n(1/\sqrt{2})$. Thus, combining Theorem 3.5 and a well known result of H. B. Lawson, which states that there exist closed orientable embedded minimal surfaces of arbitrary genus in $\mathbb{S}^3$ (see [11]), we have

THEOREM 3.7: *There exist closed orientable embedded nonminimal biharmonic surfaces of arbitrary genus in $\mathbb{S}^4$.*

This shows the existence of an abundance of biharmonic surfaces in $\mathbb{S}^4$, in contrast with the case of $\mathbb{S}^3$.

Example 3.8: To obtain a nonorientable example we consider the Veronese surface in $\mathbb{S}^4$. The map $\phi\colon \mathbb{R}^3 \to \mathbb{R}^6$ given by

$$\phi(x^1, x^2, x^3) = \Big(\frac{1}{\sqrt{6}}x^2x^3, \frac{1}{\sqrt{6}}x^1x^3, \frac{1}{\sqrt{6}}x^1x^2,$$
$$\frac{1}{2\sqrt{6}}((x^1)^2 - (x^2)^2), \frac{1}{6\sqrt{2}}((x^1)^2 + (x^2)^2 - 2(x^3)^2), 1/\sqrt{2}\Big)$$

defines a nonminimal biharmonic embedding of $P^2(\mathbb{R})$ in $\mathbb{S}^5$.

At first sight it could seem possible to construct biharmonic submanifolds in $\mathbb{S}^{n+1}$ from a nonminimal submanifold in $\mathbb{S}^n(1/\sqrt{2}) \times \{\pm 1/\sqrt{2}\}$. The following theorem shows that this is not the case.

THEOREM 3.9: *Assume that M is a submanifold in $\mathbb{S}^n(1/\sqrt{2}) \times \{\pm 1/\sqrt{2}\}$. Then M is biharmonic in $\mathbb{S}^{n+1}$ if and only if it is harmonic in $\mathbb{S}^n(1/\sqrt{2}) \times \{\pm 1/\sqrt{2}\}$.*

Proof: If $M \subset \mathbb{S}^n(a) \times \{b\}$, we have $\tau(\mathbf{i}_1 \circ \mathbf{i}) = \tau(\mathbf{i}) - \frac{m}{c}\eta \neq 0$ and

$$\tau_2(\mathbf{i}_1 \circ \mathbf{i}) = \tau_2(\mathbf{i}) + m\Big(1 - \frac{1}{a^2}\Big)\tau(\mathbf{i}) + \frac{1}{c}\Big\{|\tau(\mathbf{i})|^2 - \frac{m^2}{c^2}(c^2 - 1)\Big\}\eta. \tag{3.5}$$

When $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$ condition (3.2) reduces to

$$\tau_2(\mathbf{i}_1 \circ \mathbf{i}) = \tau_2(\mathbf{i}) - m\tau(\mathbf{i}) + |\tau(\mathbf{i})|^2\eta. \quad \blacksquare$$

The same argument as in Theorem 3.5 leads to the following

PROPOSITION 3.10: *Let M be a harmonic submanifold of $\mathbb{S}^{n_1}(r_1)$, with $0 < m < n_1$, or $M = \mathbb{S}^{n_1}(r_1)$, and let $b \in \mathbb{S}^{n_2}(r_2)$, where $r_1^2 + r_2^2 = 1$ and $n = n_1 + n_2$. Then $M \times \{b\}$ is harmonic in $\mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2)$, and nonharmonic biharmonic in $\mathbb{S}^{n+1}$ if and only if $r_1 = r_2 = 1/\sqrt{2}$.*

The biharmonic submanifolds that we have produced so far are all pseudo-umbilical. We want now to find examples of biharmonic submanifolds in $\mathbb{S}^n$ that are not of this type.

With this aim, let n_1, n_2 be two positive integers such that $n = n_1 + n_2$, and let r_1, r_2 be two positive real numbers such that $r_1^2 + r_2^2 = 1$. Let M_1 be a minimal submanifold of $\mathbb{S}^{n_1}(r_1)$ of dimension m_1, with $0 < m_1 < n_1$, and let M_2 be a minimal submanifold of $\mathbb{S}^{n_2}(r_2)$ of dimension m_2, with $0 < m_2 < n_2$. Then we have the following

THEOREM 3.11: *The manifold $M_1 \times M_2$ is a nonharmonic biharmonic submanifold of $\mathbb{S}^{n+1}$ if and only if $r_1 = r_2 = 1/\sqrt{2}$ and $m_1 \neq m_2$.*

Proof: The proof is similar to that of Theorem 3.5. ∎

Remark 3.12: When $r_1 = r_2 = 1/\sqrt{2}$ and $m_1 \neq m_2$, $M_1 \times M_2$ is not pseudo-umbilical in $\mathbb{S}^{n+1}$.

Finally, as in Theorem 3.9, we obtain

THEOREM 3.13: *Let n_1, n_2 be two positive integers such that $n = n_1 + n_2$, and let M_1 be a submanifold of $\mathbb{S}^{n_1}(1/\sqrt{2})$ of dimension m_1, with $0 < m_1 < n_1$, and let M_2 be a submanifold of $\mathbb{S}^{n_2}(1/\sqrt{2})$ of dimension m_2, with $0 < m_2 < n_2$. Then $M_1 \times M_2$ is biharmonic in $\mathbb{S}^{n+1}$ if and only if*

$$\begin{cases} \tau_2(\mathbf{i_1}) + (m_2 - m_1)\tau(\mathbf{i_1}) = 0, \\ \tau_2(\mathbf{i_2}) + (m_1 - m_2)\tau(\mathbf{i_2}) = 0, \\ |\tau(\mathbf{i_1})| = |\tau(\mathbf{i_2})|, \end{cases}$$

where $\mathbf{i_1}\colon M_1 \to \mathbb{S}^{n_1}(1/\sqrt{2})$ and $\mathbf{i_2}\colon M_2 \to \mathbb{S}^{n_2}(1/\sqrt{2})$ are the canonical inclusions.

Of course, if M_1 is harmonic in $\mathbb{S}^{n_1}(1/\sqrt{2})$, then $M_1 \times M_2$ is biharmonic in $\mathbb{S}^{n+1}$ if and only if M_2 is harmonic in $\mathbb{S}^{n_2}(1/\sqrt{2})$.

4. Biharmonic curves in $\mathbb{S}^n$

In this last section we consider biharmonic curves in $\mathbb{S}^n$. In order to derive the differential equation of nonharmonic biharmonic curves we prove the following

PROPOSITION 4.1: *Let ϕ: $(M,g) \to \mathbb{S}^n$ be a Riemannian immersion and let $\varphi = \mathbf{i} \circ \phi$, where $\mathbf{i}$: $\mathbb{S}^n \to \mathbb{R}^{n+1}$ is the canonical inclusion. Then we have*

$$\tau_2(\phi) = \tau_2(\varphi) + 2m\tau(\varphi) + \{2m^2 - |\tau(\varphi)|^2\}\varphi. \tag{4.1}$$

Proof: With respect to a system of normal coordinates with origin at an arbitrary point $p \in M$ we have, at p,

$$\tau_2(\phi) = \sum_i \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \tau(\phi) + m\tau(\phi). \tag{4.2}$$

Since $\tau(\varphi) = \tau(\phi) - m\varphi$, we obtain

$$\begin{aligned}\nabla^{\mathbb{S}^n}_{d\phi(e_i)}\tau(\phi) &= \nabla^{\mathbb{R}^{n+1}}_{d\varphi(e_i)}\tau(\phi) - B(d\phi(e_i), \tau(\phi)) \\ &= \nabla^{\mathbb{R}^{n+1}}_{d\varphi(e_i)}\tau(\phi) = \nabla^{\mathbb{R}^{n+1}}_{d\varphi(e_i)}(\tau(\varphi) + mx),\end{aligned}$$

and therefore, at p, we have

$$\begin{aligned}&\sum_i \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \tau(\phi) \\ &= \sum_i \{\nabla^{\mathbb{R}^{n+1}}_{d\varphi(e_i)} \nabla^{\mathbb{R}^{n+1}}_{d\varphi(e_i)}(\tau(\varphi) + mx) - B(d\phi(e_i), \nabla^{\mathbb{S}^n}_{d\phi(e_i)}\tau(\phi))\} \\ &= \tau_2(\varphi) + m\tau(\varphi) + \sum_i < d\phi(e_i), \nabla^{\mathbb{S}^n}_{d\phi(e_i)}\tau(\phi) > \varphi.\end{aligned}$$

But

$$\begin{aligned}\sum_i < d\phi(e_i), \nabla^{\mathbb{S}^n}_{d\phi(e_i)}\tau(\phi) > &= - |\tau(\phi)|^2 = -|\tau(\varphi) + m\varphi|^2 \\ &= - |\tau(\varphi)|^2 + m^2.\end{aligned}$$

Hence

$$\sum_i \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \tau(\phi) = \tau_2(\varphi) + m\tau(\varphi) + (-|\tau(\varphi)|^2 + m^2)\varphi.$$

Now we replace the value of $\sum_i \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \nabla^{\mathbb{S}^n}_{d\phi(e_i)} \tau(\phi)$ in (4.2), and we obtain (4.1). ∎

When M is a curve, the biharmonic equation given by the vanishing of (4.1) gives the desired differential equation.

COROLLARY 4.2: *Let γ: $I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a curve parametrized by arc length. Then γ is biharmonic in $\mathbb{S}^n$ if and only if*

$$\gamma^{\mathrm{IV}} + 2\gamma'' + (1 - k_g^2)\gamma = 0, \tag{4.3}$$

where $k_g^2 = |\nabla_{\gamma'}^{\mathbb{S}^n}\gamma'|^2$ is the square of the geodesic curvature of γ in $\mathbb{S}^n$.

Equation (4.3) can be integrated. This is because the geodesic curvature k_g of γ is constant.

PROPOSITION 4.3: *Let $\gamma\colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a nonharmonic biharmonic curve parametrized by arc length. Then k_g is constant and $0 < |k_g| \le 1$.*

Proof: From the second equation of (3.1) we obtain that k_g is constant. To prove that $0 < |k_g| \le 1$, we put $T = \gamma'$ and $\tau(\gamma) = \nabla_T^{\mathbb{S}^n} T = k_g N$. Then $|T| = 1$ and $< N, T >= 0$. Also, let $\nabla_T^{\mathbb{S}^n} N = fT + W$, where W is a vector field along γ such that $< W, T >= 0$, and $f \in C^\infty(I)$. Then $f = -k_g$. Next, from

$$\nabla_T^{\mathbb{S}^n}\tau(\gamma) = -k_g^2 T + k_g W,$$

it follows that $A_{\tau(\gamma)}(T) = k_g^2 T$, where A is the shape operator, and $\nabla_T^{\perp}\tau(\gamma) = k_g W$. Since γ is biharmonic, we obtain

$$\Delta^{\perp}\tau(\gamma) = \tau(\gamma) - B(T, A_{\tau(\gamma)}(T)) = k_g(1 - k_g^2)N.$$

Now, from the Weitzenböck formula

$$\frac{1}{2}\Delta^{\perp}|\tau(\gamma)|^2 =< \Delta^{\perp}\tau(\gamma), \tau(\gamma) > -|\nabla^{\perp}\tau(\gamma)|^2,$$

we get $1 - k_g^2 = |W|^2$, and this completes the proof. ∎

Since k_g is constant, integration of (4.3) is possible and it yields the following

PROPOSITION 4.4: *Let $\gamma\colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a curve parametrized by arc length. We have two classes of nonharmonic biharmonic curves in $\mathbb{S}^n$.*

1. *When $k_g = 1$ they are circles parametrized by*

$$\gamma(t) = \cos(\sqrt{2}t)c_1 + \sin(\sqrt{2}t)c_2 + c_4,$$

where c_1, c_2, c_4 are constant vectors orthogonal to each other with $|c_1|^2 = |c_2|^2 = |c_4|^2 = \frac{1}{2}$.

2. *When $0 < k_g < 1$ they are curves that, following [7], we shall call helices, parametrized by*

$$\gamma(t) = \cos(at)c_1 + \sin(bt)c_2 + \cos(bt)c_3 + \sin(bt)c_4,$$

where c_1, c_2, c_3 and c_4 are constant vectors orthogonal to each other with $|c_1|^2 = |c_2|^2 = |c_3|^2 = |c_4|^2 = \frac{1}{2}$, and $a^2 + b^2 = 2$, $a^2 \ne b^2$. In this case $k_g^2 = 1 - a^2b^2 \in (0,1)$.

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